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Infinite mode analysis and truncation to resonant modes of axially accelerated beam vibrations

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Abstract

The transverse vibrations of simply supported axially moving Euler–Bernoulli beams are investigated. The beam has a time-varying axial velocity with viscous damping. Traveling beam eigenfunctions with infinite number of modes are considered. Approximate analytical solutions are sought using the method of Multiple Scales, a perturbation technique. A detailed analysis of the resonances in which upto four modes of vibration involved are performed. Stability analysis is treated for each type of resonance. Approximate stability borders are given for the resonances involving only two modes. For higher number of modes involved in a resonance, sample numerical examples are presented for stabilities.

1. Introduction

Band-saws, chains, conveyer belts, fiber textiles, magnetic tapes, paper sheets, aerial tramways, pipes transporting fluids, threadlines are some technological examples of problems classified as axially moving continua. Either a string model or a beam model can be used to analyze such systems. Linear and nonlinear models exist in the literature. For a review of the vast literature, see papers by Ulsoy et al. [1], Wickert and Mote [2] and Chen [3]. The axially moving strings and beams in transverse motions were investigated and the effect of tension was included by Wickert and Mote [4]. Wickert [5] included the nonlinear stretching effects for a moving beam and analyzed both the subcritical and supercritical speed range for a traveling beam. Using a similar model, Chakraborty et al. [6] investigated the free and forced responses of a traveling beam.

While most of the studies deal with the constant velocity problem, some papers addressed the influence of speed variation on the vibrations. Mote [7] investigated the problem of an axially accelerating string with harmonic excitation at one end; he replaced the variable coefficients by their time averaged values and investigated stability by Laplace transform techniques. Pakdemirli et al. [8] used Floquet theory to determine stability of a string moving with harmonically varying velocity. A constant acceleration–deceleration type periodic velocity case has also been investigated [9]. Using the asymptotic approach of Krylov, Bogoliubov and Mitropolsky, Wickert [10] developed the amplitude and phase modulations for an accelerating string.

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Pakdemirli and Ulsoy [11] used the method of multiple scales to predict stability of a string moving with a harmonically varying velocity. Traveling string eigenfunctions are not orthogonal to each other and this appears as a serious problem in discretization. To eliminate the problem, Wickert and Mote [12,13] transferred the equations of motion into a first-order system and selected suitable orthogonal basis functions to discretize the equations. In the direct perturbation method, however, the method of multiple scales is applied directly to the partial differential equation and this method does not require transformation of the equations or the selection of an orthonormal basis (see Ref. [11] for comparisons of both approaches). Öz and Pakdemirli [14] and Öz [15] extended the analysis in Ref. [11] to beams for different end conditions.

Recently, some papers investigated the transition behavior from string to beam for axially moving materials. Since the flexural stiffness term, which contains the highest order derivative is small compared to other terms, the problem is essentially a boundary layer problem. Öz et al. [16] constructed an outer solution for a variable speed beam using the method of multiple scales. This solution does not satisfy the moment conditions at the ends. Pellicano and Zirilli [17] constructed a boundary layer solution for constant velocity case using a linear and nonlinear model. Pakdemirli and Özkaya [18] used the method of multiple scales and found a boundary layer solution for constant velocity case.

Variable axial transport velocity problem has been addressed in some recent work [19–28]. Viscoelastic string and beam models with harmonically varying axially transport velocity were considered by Chen et al. [19,20], Chen and Yang [21,22], Yang and Chen [23], Chen [24] and Chen and Yang [25]. Comparison of different mathematical models, comparison and employment of different numerical and analytical techniques were discussed in the mentioned papers. Simply supported beam with torsional springs at the supports were considered in Ref. [25]. In all papers addressing speed fluctuations, parallel to the applications, the mean speed was taken to be of order 1 and the variations in the speed to be of order ε, ε representing a much smaller quantity than 1. On the other hand, Suweken and Horssen [26,27] considered a type of harmonic variation in which the mean speed and variations in velocity were both of the same order and small. This selection has limited practical application. Moreover, stationary string eigenfunctions used throughout the analysis have poor convergence properties. For a vanishing mean velocity, they might be used as was done in Refs. [26,27] but for a large mean velocity, traveling eigenfunctions are much better. The truncation to finite number of modes were addressed in detail claiming that the truncation leads to incorrect solutions. Truncated mode solutions can be considered as solutions valid if all resonant modes are taken into account and non-resonant modes ignored. For this to happen, the system should be non-conservative and in real systems one always have a damping mechanism which enables decaying of non-resonant modes. In axially moving strings, the natural frequencies appear as integer multiples of each other yielding infinitely many resonance conditions. The problem was addressed in a recent paper by Ponomavera and Van Horssen [28]. Their system was conservative without damping and the mean velocities are of order 1 compared to the velocity fluctuations which are of order ε. Some calculations for the amplitudes of vibrations were presented for the infinite mode interaction case but as expected, no applicable solution was presented for the general case of infinite mode resonances. They also mentioned that extracting more information from their system of equations (i.e. Eqs. (33) and (34)) remains an open subject for future research. For the beam problem considered here, frequencies are not exact multiples of each other and resonances are not expected to be as rich as the string case. However, special care has to be taken in identifying all resonant modes and this requires information and therefore numerical calculation of a large number of modes. Some comments on this issue will be given in the subsequent chapters.

In this work, we generalize our previous work (with emphasis to Ref. [14]) with finite number of modes to an analysis which can be applied to infinite number of modes. To better represent real systems, a damping term is also included to deviate the system form a conservative one to a more realistic non-conservative one. Mode interactions are studied in detail. In addition to the principal parametric resonances, sum- and difference-type resonances of the previous work [14], more interactions involving upto four different modes are considered. Mode interactions depend on the specific numerical values of the modes which are functions of mean velocity and flexural stiffness and it is shown that under certain conditions, the modes that are not excited through some mechanism of the fluctuation frequency decay in time. For some specific parameters, resonances involving more than four modes may appear. This requires numerical calculation of infinite number of modes which is practically impossible because modes are also affected by the mean velocity and stiffness of the system. Instead, the analysis is limited to four mode interactions at most and parameters such as fluctuation frequency and mean velocity are selected to avoid higher number of mode interactions. A detailed stability analysis corresponding to each class of interaction is given. Numerical examples of different cases are presented.

Finally, vibrations of pipes excited by conveying fluids problem is mathematically similar to the axially moving continua problem. A review of the vast literature is beyond the scope of this work. The reader is referred to the comprehensive review paper by Paidoussis and Li [29]. When the fluid velocity is harmonically varying with time about a constant mean velocity, and the pipe stiffness cannot be ignored, the problem becomes similar to the problem considered here. See Semler and Paidoussis [30] for example.

2. Equations of motion

The equations of motion for a moving beam with variable axial transport velocity $\upsilon(t)$ is [14]

$$\ddot{u} + \bar{\mu}(\dot{u} + \upsilon u') + 2\upsilon \dot{u}' + \dot{\upsilon} u' + (\upsilon^2 - 1)u'' + \upsilon_f^2 u^{iv} = 0, \tag{1}$$

where u denotes the transverse deflection, dot denotes derivative with respect to time (t) and prime denotes derivative with respect to the spatial derivative (x). The equation of motion is in dimensionless form. Dimensionless quantities are defined through the relations

$$u = \frac{u^*}{L},\ \ x = \frac{x^*}{L},\ \ t = t^*\sqrt{P/\rho A L^2},\ \ \upsilon = \frac{\upsilon^*}{\sqrt{P/\rho A}},\ \ \upsilon_f = \sqrt{EI/PL^2}, \tag{2}$$

where variables with asterisk denote dimensional ones. P is the axial tension force, ρ the density, A the cross-sectional area, L the length and EI the flexural rigidity and $\bar{\mu}$ the damping coefficient of the beam.

Simply supported beams will be considered throughout the analysis (see Fig. 1)

$$u(0,t) = 0,\ \ u(1,t) = 0,\ \ u''(0,t) = 0,\ \ u''(1,t) = 0. \tag{3}$$

For speed variation, a harmonic model with small fluctuations about a constant mean velocity is assumed

$$\upsilon = \upsilon_0 + \varepsilon \upsilon_1 \sin \Omega t, \tag{4}$$

where $\varepsilon\upsilon_1$ is the amplitude and Ω is the frequency of fluctuations. ε is a small parameter to ensure that harmonic variations are small compared to the mean velocity υ_0. Note that the second term in the model (i.e. Eq. (1)) is the damping term to make the system non-conservative. In axially moving materials due to the combination of axial velocity and vertical displacement rate, the damping model is usually chosen as given above.

3. Perturbation solution

In this section, an approximate solution will be sought using a special perturbation technique, namely the method of multiple scales [31,32]. This method will be applied directly to the partial differential equation (Direct perturbation method). Direct perturbation method has advantages over the discretization–perturbation method [33–38]. In higher order schemes and for finite mode truncations, the method yields better approximations to the real problem. For the special case considered here, the advantage is that, there is no need to cast equation of motion into a convenient first-order form where orthogonality of traveling string eigenfunctions is achieved as was done by Wickert and Mote [12,13].

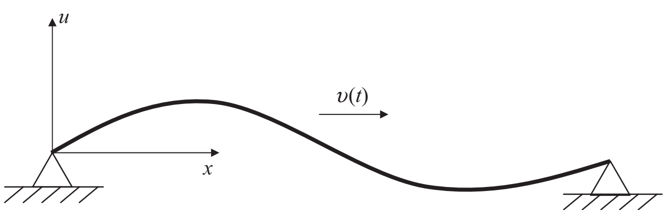

Fig. 1. Schematics of an axially moving simply supported beam.

In the direct method, one assumes an expansion of the form

$$u(x,t;\varepsilon) = u_0(x,T_0,T_1) + \varepsilon u_1(x,T_0,T_1) + \cdots, \tag{5}$$

where $T_0 = t$ and $T_1 = \varepsilon t$ are the usual fast and slow time scales of the Method of Multiple Scales. Time derivatives are defined as

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{6}$$

where $D_i = \partial/\partial T_i$. For the light damping in such systems $\bar{\mu} = \varepsilon\mu$ can be selected. Substituting Eqs. (4)–(6) into the partial differential system (1) and (3), seperating terms at each order of ε, one has

$$\begin{aligned} O(1):\quad & D_0^2 u_0 + 2\upsilon_0 D_0 u_0' + (\upsilon_0^2 - 1)u_0'' + \upsilon_f^2 u_0^{iv} = 0, \\ & u_0(0,t) = u_0(1,t) = u_0''(0,t) = u_0''(1,t) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} O(\varepsilon):\quad & D_0^2 u_1 + 2\upsilon_0 D_0 u_1' + (\upsilon_0^2 - 1)u_1'' + \upsilon_f^2 u_1^{iv} = -2D_0 D_1 u_0 - \mu(D_0 u_0 + \upsilon_0 u_0') \\ & - 2\upsilon_0 D_1 u_0' - 2\upsilon_1 \sin \Omega T_0 D_0 u_0' - \upsilon_1 \Omega \cos \Omega T_0 u_0' - 2\upsilon_0\upsilon_1 \sin \Omega T_0 u_0'', \\ & u_1(0,t) = u_1(1,t) = u_1''(0,t) = u_1''(1,t) = 0. \end{aligned} \tag{8}$$

At order 1, including all modes of vibration, the solution is

$$u_0 = \sum_{m=1}^{\infty} (A_m(T_1)\mathrm{e}^{\mathrm{i}\omega_m T_0}\psi_m(x) + \overline{A}_m(T_1)\mathrm{e}^{-\mathrm{i}\omega_m T_0}\bar{\psi}_m(x)), \tag{9}$$

where ω_m represent the natural frequencies. The mode shapes ψ_m were calculated previously [14,15]

$$\begin{aligned} \psi_m(x) = c_1 \Bigg\{ & \mathrm{e}^{\mathrm{i}\beta_{1m}x} - \frac{(\beta_{4m}^2 - \beta_{1m}^2)(\mathrm{e}^{\mathrm{i}\beta_{3m}} - \mathrm{e}^{\mathrm{i}\beta_{1m}})}{(\beta_{4m}^2 - \beta_{2m}^2)(\mathrm{e}^{\mathrm{i}\beta_{3m}} - \mathrm{e}^{\mathrm{i}\beta_{2m}})}\mathrm{e}^{\mathrm{i}\beta_{2m}x} - \frac{(\beta_{4m}^2 - \beta_{1m}^2)(\mathrm{e}^{\mathrm{i}\beta_{2m}} - \mathrm{e}^{\mathrm{i}\beta_{1m}})}{(\beta_{4m}^2 - \beta_{3m}^2)(\mathrm{e}^{\mathrm{i}\beta_{2m}} - \mathrm{e}^{\mathrm{i}\beta_{3m}})}\mathrm{e}^{\mathrm{i}\beta_{3m}x} \\ & + \left[-1 + \frac{(\beta_{4m}^2 - \beta_{1m}^2)(\mathrm{e}^{\mathrm{i}\beta_{3m}} - \mathrm{e}^{\mathrm{i}\beta_{1m}})}{(\beta_{4m}^2 - \beta_{2m}^2)(\mathrm{e}^{\mathrm{i}\beta_{3m}} - \mathrm{e}^{\mathrm{i}\beta_{2m}})} + \frac{(\beta_{4m}^2 - \beta_{1m}^2)(\mathrm{e}^{\mathrm{i}\beta_{2m}} - \mathrm{e}^{\mathrm{i}\beta_{1m}})}{(\beta_{4m}^2 - \beta_{3m}^2)(\mathrm{e}^{\mathrm{i}\beta_{2m}} - \mathrm{e}^{\mathrm{i}\beta_{3m}})} \right] \mathrm{e}^{\mathrm{i}\beta_{4m}x} \Bigg\}, \end{aligned} \tag{10}$$

where β_{im} are eigenvalues. The eigenvalues were obtained previously by using the following frequency equation and support condition [14,15]:

$$\upsilon_f^2 \beta_{im}^4 + (1 - \upsilon_0^2)\beta_{im}^2 - 2\omega_m \upsilon_0 \beta_{im} - \omega_m^2 = 0, \quad i = 1,2,3,4, \quad m = 1,2\ldots, \tag{11}$$

$$\begin{aligned} & [\mathrm{e}^{\mathrm{i}(\beta_{1m}+\beta_{2m})} + \mathrm{e}^{\mathrm{i}(\beta_{3m}+\beta_{4m})}](\beta_{1m}^2 - \beta_{2m}^2)(\beta_{3m}^2 - \beta_{4m}^2) + [\mathrm{e}^{\mathrm{i}(\beta_{1m}+\beta_{3m})} + \mathrm{e}^{\mathrm{i}(\beta_{2m}+\beta_{4m})}](\beta_{2m}^2 - \beta_{4m}^2)(\beta_{3m}^2 - \beta_{1m}^2) \\ & + [\mathrm{e}^{\mathrm{i}(\beta_{2m}+\beta_{3m})} + \mathrm{e}^{\mathrm{i}(\beta_{1m}+\beta_{4m})}](\beta_{1m}^2 - \beta_{4m}^2)(\beta_{2m}^2 - \beta_{3m}^2) = 0. \end{aligned} \tag{12}$$

Contrary to the string case, natural frequencies of beam case treated here can only be calculated numerically. The frequencies and resonance conditions are discussed in the numerical analysis section. Eqs. (10)–(12) can be retrieved from the ones presented in Ref. [25] in the limiting case of vanishing torsional springs.

At order ε, inserting Eq. (9) into Eq. (8) and arranging, one has

$$\begin{aligned} & D_0^2 u_1 + 2\upsilon_0 D_0 u_1' + (\upsilon_0^2 - 1)u_1'' + \upsilon_f^2 u_1^{iv} \\ & = \sum_{m=1}^{\infty} \Bigg\{ D_1 A_m(-2\mathrm{i}\omega_m \psi_m - 2\upsilon_0 \psi_m')\mathrm{e}^{\mathrm{i}\omega_m T_0} - \mu A_m(\mathrm{i}\omega_m \psi_m + \upsilon_0 \psi_m')\mathrm{e}^{\mathrm{i}\omega_m T_0} \\ & + \left(-\upsilon_1 \omega_m \psi_m' - \frac{\upsilon_1 \Omega}{2}\psi_m' + \mathrm{i}\upsilon_0\upsilon_1 \psi_m'' \right) A_m \mathrm{e}^{\mathrm{i}(\Omega+\omega_m)T_0} \\ & + \left(\upsilon_1 \omega_m \bar{\psi}_m' - \frac{\upsilon_1 \Omega}{2}\bar{\psi}_m' + \mathrm{i}\upsilon_0\upsilon_1 \bar{\psi}_m'' \right) \overline{A}_m \mathrm{e}^{\mathrm{i}(\Omega-\omega_m)T_0} \Bigg\} + \mathrm{cc}, \end{aligned} \tag{13}$$

where cc stands for complex conjugates of the preceeding terms.

A solution of the form given below can be proposed

$$u_1 = \sum_{n=1}^{\infty}(Y_n(x, T_1)e^{i\omega_n T_0} + W_n(x, T_0, T_1) + cc). \tag{14}$$

Substituting this solution into Eq. (13) and rearranging yields

$$\begin{aligned}
&\sum_{n=1}^{\infty}[-\omega_n^2 Y_n + 2\upsilon_0 i\omega_n Y_n' + v_f^2 Y_n^{iv} + (\upsilon_0^2 - 1)Y_n'']e^{i\omega_n T_0} \\
&\quad + D_0^2 W_n + 2\upsilon_0 D_0 W_n' + v_f^2 W_n^{iv} + (\upsilon_0^2 - 1)W_n'' + cc \\
&= \sum_{m=1}^{\infty}\Bigg\{ D_1 A_m(-2i\omega_m\psi_m - 2\upsilon_0\psi_m')e^{i\omega_m T_0} - \mu A_m(i\omega_m\psi_m + \upsilon_0\psi_m')e^{i\omega_m T_0} \\
&\quad + \left(-\upsilon_1\omega_m\psi_m' - \frac{\upsilon_1\Omega}{2}\psi_m' + i\upsilon_0\upsilon_1\psi_m''\right)A_m e^{i(\Omega+\omega_m)T_0} \\
&\quad + \left(\upsilon_1\omega_m\bar{\psi}_m' - \frac{\upsilon_1\Omega}{2}\bar{\psi}_m' + i\upsilon_0\upsilon_1\bar{\psi}_m''\right)\overline{A}_m e^{i(\Omega-\omega_m)T_0}\Bigg\} + cc.
\end{aligned} \tag{15}$$

No assumptions for mode truncations are made in obtaining the above equation. Infinite modes are taken in the analysis. Therefore Eq. (15) posseses all possible resonances inherited in the system. This equation forms the basis of our analysis. Under some assumptions for mode interactions, the equation seperates and simplifies. As will be mentioned later, resonances depend on the numerical values of natural frequencies and the numerical value of velocity fluctuation frequency Ω. Four distinct cases with some subcases can now be considered and with regard to the specific selection, different equations are retrieved. Upto four mode interactions will be considered in the subsequent analysis. Higher number of mode interactions might be possible depending on the numerical value of fluctuation frequency and natural frequencies but numerical and stability analysis of such complex interactions will not be considered in this analysis and will be left to further studies.

3.1. No resonance case ($\Omega+\omega_m \ncong \omega_n$, $\Omega-\omega_m \ncong \omega_n$)

For this specific choice, seperating Eq. (15) into secular and non-secular terms and imposing the solvability condition [31] on the secular terms yields evolution equation of complex amplitudes

$$2D_1 A_n + \mu A_n = 0. \tag{16}$$

The solution is

$$A_n = A_0 e^{-(\mu/2)T_1}. \tag{17}$$

This solution indicates that all modes that are not parametrically excited would decay in time. Details of solvability calculations on a similar problem can be found in Ref. [11].

3.2. Sum-type combination resonances only ($\Omega+\omega_m \ncong \omega_n$, $\Omega-\omega_m \cong \omega_n$)

In this case $\Omega \cong \omega_m + \omega_n$ and the resonances are only sum-type combination resonances. To express the nearness of fluctuation frequency to the sum of any nth and mth modes, one may select

$$\Omega = \omega_m + \omega_n + \varepsilon\sigma, \tag{18}$$

where σ is a detuning parameter. For this choice, similar calculations with the previous case yields the solvability conditions:

$$D_1 A_n + \frac{\mu}{2}A_n + k_{3mn}\overline{A}_m e^{i\sigma T_1} = 0, \tag{19}$$

$$D_1 A_m + \frac{\mu}{2} A_m + k_{4mn} \overline{A}_n \mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{20}$$

where the coefficients k_{3mn} and k_{4mn} are defined as follows:

$$k_{3mn} = -\frac{\int_0^1 (\upsilon_1 \omega_m \bar{\psi}'_m - (\upsilon_1 \Omega/2)\bar{\psi}'_m + \mathrm{i}\upsilon_0\upsilon_1 \bar{\psi}''_m)\bar{\psi}_n \,\mathrm{d}x}{2\int_0^1 (\mathrm{i}\omega_n \psi_n + \upsilon_0 \psi'_n)\bar{\psi}_n \,\mathrm{d}x}, \tag{21}$$

$$k_{4mn} = -\frac{\int_0^1 (\upsilon_1 \omega_n \bar{\psi}'_n - (\upsilon_1 \Omega/2)\bar{\psi}'_n + \mathrm{i}\upsilon_0\upsilon_1 \bar{\psi}''_n)\bar{\psi}_m \,\mathrm{d}x}{2\int_0^1 (\mathrm{i}\omega_m \psi_m + \upsilon_0 \psi'_m)\bar{\psi}_m \,\mathrm{d}x}. \tag{22}$$

Note that $k_{4mn} = k_{3nm}$. A special subcase is discussed below.

3.2.1. Special sum-type resonances (principal parametric resonances) ($\Omega+\omega_n \ncong \omega_n$, $\Omega-\omega_n \cong \omega_n$)

When $m = n$, then $\Omega \cong 2\omega_n$ and a special type of sum-type resonance namely the principal parametric resonance occurs. For this degenerate case, the nearness of fluctuation frequency to twice a natural frequency is expressed as

$$\Omega = 2\omega_n + \varepsilon\sigma \tag{23}$$

and Eqs. (19)–(22) reduce to

$$D_1 A_n + \frac{\mu}{2} A_n + k_{0n} \overline{A}_n \mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{24}$$

where

$$k_{0n} = -\frac{\int_0^1 (\upsilon_1 \omega_n \bar{\psi}'_n - (\upsilon_1 \Omega/2)\bar{\psi}'_n + \mathrm{i}\upsilon_0\upsilon_1 \bar{\psi}''_n)\bar{\psi}_n \,\mathrm{d}x}{2\int_0^1 (\mathrm{i}\omega_n \psi_n + \upsilon_0 \psi'_n)\bar{\psi}_n \,\mathrm{d}x}. \tag{25}$$

3.3. Difference-type combination resonances only ($\Omega+\omega_m \cong \omega_n$, $\Omega-\omega_m \ncong \omega_n$)

For this special case, one may assume that there is only a combination resonance of difference-type between the nth and mth modes, that is $\Omega \cong \omega_n - \omega_m$. Without loss of generality one may assume $\omega_n > \omega_m$ and the nearness of fluctuation frequency to the difference of any two modes may be expressed as

$$\Omega = \omega_n - \omega_m + \varepsilon\sigma. \tag{26}$$

With reference to Eq. (15), for this special choice, the solvability conditions are obtained:

$$D_1 A_n + \frac{\mu}{2} A_n + k_{5mn} A_m \mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{27}$$

$$D_1 A_m + \frac{\mu}{2} A_m + k_{6mn} A_n \mathrm{e}^{-\mathrm{i}\sigma T_1} = 0, \tag{28}$$

where the coefficients are defined as follows:

$$k_{5mn} = -\frac{\int_0^1 (-\upsilon_1 \omega_m \psi'_m - (\upsilon_1 \Omega/2)\psi'_m + \mathrm{i}\upsilon_0\upsilon_1 \psi''_m)\bar{\psi}_n \,\mathrm{d}x}{2\int_0^1 (\mathrm{i}\omega_n \psi_n + \upsilon_0 \psi'_n)\bar{\psi}_n \,\mathrm{d}x}, \tag{29}$$

$$k_{6mn} = -\frac{\int_0^1 (\upsilon_1 \omega_n \psi'_n - (\upsilon_1 \Omega/2)\psi'_n - \mathrm{i}\upsilon_0\upsilon_1 \psi''_n)\bar{\psi}_m \,\mathrm{d}x}{2\int_0^1 (\mathrm{i}\omega_m \psi_m + \upsilon_0 \psi'_m)\bar{\psi}_m \,\mathrm{d}x}. \tag{30}$$

3.3.1. Special difference-type resonances ($\Omega+\omega_n \cong \omega_n$, $\Omega-\omega_n \ncong \omega_n$)

When $m = n$, then $\Omega \cong 0$ and a special type of difference-type resonance occurs. This case corresponds to very slow changes in axial transport velocity. For this degenerate case, the vanishing fluctuation frequency can be expressed as

$$\Omega = \varepsilon\sigma \tag{31}$$

and Eqs. (27)–(30) reduce to

$$D_1 A_n + \frac{\mu}{2} A_n + (k_{7n} \cos \sigma T_1 + k_{8n} \sin \sigma T_1) A_n = 0, \tag{32}$$

where

$$k_{7n} = \frac{\int_0^1 \upsilon_1 \Omega \psi_n' \bar{\psi}_n \, \mathrm{d}x}{2 \int_0^1 (\mathrm{i}\omega_n \psi_n + \upsilon_0 \psi_n') \bar{\psi}_n \, \mathrm{d}x}, \tag{33}$$

$$k_{8n} = \frac{\int_0^1 (\mathrm{i}\upsilon_1 \omega_n \psi_n' + \upsilon_0 \upsilon_1 \psi_n'') \bar{\psi}_n \, \mathrm{d}x}{\int_0^1 (\mathrm{i}\omega_n \psi_n + \upsilon_0 \psi_n') \bar{\psi}_n \, \mathrm{d}x}. \tag{34}$$

3.4. Sum- and difference-type combination resonances together ($\Omega+\omega_p \cong \omega_q$, $\Omega-\omega_m \cong \omega_n$) ($m \neq n \neq p \neq q$)

In a more general case, sum- and difference-type resonances both occur simultaneously and four different modes are involved in the resonances. The nearness of fluctuation frequency to those modes are then expressed as follows:

$$\begin{aligned} \Omega &= \omega_n + \omega_m + \varepsilon\sigma_1 \quad \text{(sum type)}, \\ \Omega &= \omega_q - \omega_p + \varepsilon\sigma_2 \quad \text{(difference type)}. \end{aligned} \tag{35}$$

Complex amplitudes A_m and A_n are coupled. Complex amplitudes A_p and A_q are also coupled. But there is no coupling between (m,n) and (p,q) pairs. Therefore stability can be investigated independently. Section 3.2 and 3.3 equations apply for each case. The equations are:

$$\begin{aligned} &D_1 A_n + \frac{\mu}{2} A_n + k_{3mn} \overline{A}_m \mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0, \\ &D_1 A_m + \frac{\mu}{2} A_m + k_{3nm} \overline{A}_n \mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0, \\ &D_1 A_q + \frac{\mu}{2} A_q + k_{5pq} A_p \mathrm{e}^{\mathrm{i}\sigma_2 T_1} = 0, \\ &D_1 A_p + \frac{\mu}{2} A_p + k_{6pq} A_q \mathrm{e}^{-\mathrm{i}\sigma_2 T_1} = 0. \end{aligned} \tag{36}$$

This is the general case. Several degenerate subcases exist which needs special treatment.

3.4.1. $n = q$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_p \cong \omega_n$)

Note that for coupling between (m,n) and (p,q) modes $n = q$ condition will lead to $\Omega-\omega_m \cong \Omega+\omega_p$ or $\omega_p \cong -\omega_m$ which cannot occur since natural frequencies are always postive numbers.

3.4.2. $m = q$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_p \cong \omega_m$)

In this case $\Omega \cong \omega_m+\omega_n$ and $\Omega \cong \omega_m-\omega_p$ can be written. For a given Ω value, this case is impossible also. Equating $\Omega \cong \omega_m+\omega_n \cong \omega_m-\omega_p$ yields $\omega_n \cong -\omega_p$ which is impossible.

3.4.3. $n = p$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_n \cong \omega_q$)

In this case, $\Omega \cong \omega_m+\omega_n$, $\Omega \cong \omega_q-\omega_n$ can be written. The equations are:

$$\begin{aligned} &D_1 A_n + \frac{\mu}{2} A_n + k_{3mn} \overline{A}_m \mathrm{e}^{\mathrm{i}\sigma_1 T_1} + k_{6nq} A_q \mathrm{e}^{-\mathrm{i}\sigma_2 T_1} = 0, \\ &D_1 A_m + \frac{\mu}{2} A_m + k_{3nm} \overline{A}_n \mathrm{e}^{\mathrm{i}\sigma_1 T_1} = 0, \\ &D_1 A_q + \frac{\mu}{2} A_q + k_{5nq} A_n \mathrm{e}^{\mathrm{i}\sigma_2 T_1} = 0. \end{aligned} \tag{37}$$

3.4.4. $m=p$ case ($\Omega-\omega_m\cong\omega_n$, $\Omega+\omega_m\cong\omega_q$)

In this case, $\Omega\cong\omega_m+\omega_n$, $\Omega\cong\omega_q-\omega_m$ can be written. The equations are:

$$\begin{aligned} &D_1A_n+\frac{\mu}{2}A_n+k_{3mn}\overline{A}_m\mathrm{e}^{\mathrm{i}\sigma_1T_1}=0,\\ &D_1A_m+\frac{\mu}{2}A_m+k_{3nm}\overline{A}_n\mathrm{e}^{\mathrm{i}\sigma_1T_1}+k_{6mq}A_q\mathrm{e}^{-\mathrm{i}\sigma_2T_1}=0,\\ &D_1A_q+\frac{\mu}{2}A_q+k_{5mq}A_m\mathrm{e}^{\mathrm{i}\sigma_2T_1}=0. \end{aligned} \tag{38}$$

This case is indeed similar to the case given in Section 3.4.3. Interchanging m with n yields exactly the same equations.

3.4.5. $m=n$ case ($\Omega-\omega_n\cong\omega_n$, $\Omega+\omega_p\cong\omega_q$)

In this case, $\Omega\cong2\omega_n$, $\Omega\cong\omega_q-\omega_p$ can be written. The equations are:

$$\begin{aligned} &D_1A_n+\frac{\mu}{2}A_n+k_{3nn}\overline{A}_n\mathrm{e}^{\mathrm{i}\sigma_1T_1}=0,\\ &D_1A_q+\frac{\mu}{2}A_q+k_{5pq}A_p\mathrm{e}^{\mathrm{i}\sigma_2T_1}=0,\\ &D_1A_p+\frac{\mu}{2}A_p+k_{6pq}A_q\mathrm{e}^{-\mathrm{i}\sigma_2T_1}=0. \end{aligned} \tag{39}$$

Note that A_p and A_q are uncoupled from A_n.

3.4.6. $p=q$ case ($\Omega+\omega_p\cong\omega_p$, $\Omega-\omega_m\cong\omega_n$)

In this case $\Omega\cong0$, $\Omega\cong\omega_n+\omega_m$ is obtained. However, this case is impossible and hence discarded.

3.4.7. $n=p=m$ case ($\Omega-\omega_n\cong\omega_n$, $\Omega+\omega_n\cong\omega_q$)

This case is a further subcase of 3.4.3 and has to be treated seperately. In this case, $\Omega\cong2\omega_n$, $\Omega\cong\omega_q-\omega_n$ can be written. The equations are:

$$\begin{aligned} &D_1A_n+\frac{\mu}{2}A_n+k_{3nn}\overline{A}_n\mathrm{e}^{\mathrm{i}\sigma_1T_1}+k_{6nq}A_q\mathrm{e}^{-\mathrm{i}\sigma_2T_1}=0,\\ &D_1A_q+\frac{\mu}{2}A_q+k_{5nq}A_n\mathrm{e}^{\mathrm{i}\sigma_2T_1}=0. \end{aligned} \tag{40}$$

Based on the equations derived in this section, a stability analysis will be performed in the next section. Note that $n=m=q$, $n=p=q$, $m=p=q$ cases all lead to impossible choices and discarded.

4. Stability analysis

In the previous section, some possible resonance cases involving upto four mode interactions and corresponding equations in terms of the evolution of complex amplitudes are given. The corresponding stability analysis for each case will be developed in this section in the same order.

4.1. No resonance case ($\Omega+\omega_m\ncong\omega_n$, $\Omega-\omega_m\ncong\omega_n$)

As indicated previously, the complex amplitudes decay in time and there is no instability.

4.2. Sum-type combination resonances only ($\Omega+\omega_m\ncong\omega_n$, $\Omega-\omega_m\cong\omega_n$)

For sum-type resonances only, the equations to be referred are Eqs. (19) and (20) in Section 3.2. A transformation of the complex amplitudes would be helpful:

$$A_n=B_n\,\mathrm{e}^{\mathrm{i}\sigma T_1/2},\quad A_m=B_m\,\mathrm{e}^{\mathrm{i}\sigma T_1/2}. \tag{41}$$

Inserting the above transformations into Eqs. (19) and (20) yields

$$D_1 B_n + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma}{2}\right) B_n + k_{3mn}\overline{B}_m = 0,$$
$$D_1 B_m + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma}{2}\right) B_m + k_{3nm}\overline{B}_n = 0. \tag{42}$$

Determining stability with regard to B_n would be equivalent to determining the stability with regard to A_n. One may introduce the below forms for B_n and B_m as

$$B_n = b_n \mathrm{e}^{\lambda T_1}, \quad B_m = b_m \mathrm{e}^{\bar{\lambda} T_1}, \tag{43}$$

where b_n and b_m are real now. Substituting Eq. (43) into Eq. (42), one gets the following matrix equation:

$$\begin{bmatrix} \lambda + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma}{2}\right) & k_{3mn} \\ \overline{k}_{3nm} & \lambda + \left(\frac{\mu}{2} - \mathrm{i}\frac{\sigma}{2}\right) \end{bmatrix} \begin{bmatrix} b_n \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{44}$$

Determinant of the coefficient matrix should be zero for non-trivial solutions and equating the determinant to zero gives the eigenvalues

$$\lambda_{1,2} = -\tfrac{1}{2}\mu \mp \tfrac{1}{2}\sqrt{-\sigma^2 + 4k_{3mn}\overline{k}_{3nm}}. \tag{45}$$

If $k_{3mn}\overline{k}_{3nm} \leqslant 0$, the system is always stable, and if $k_{3mn}\overline{k}_{3nm} > 0$ then the stability boundaries are determined by the following equation:

$$\sigma = \mp\sqrt{-\mu^2 + 4k_{3mn}\overline{k}_{3nm}} \tag{46}$$

or

$$\Omega = \omega_m + \omega_n \mp \varepsilon\sqrt{-\mu^2 + 4k_{3mn}\bar{k}_{3nm}}. \tag{47}$$

A similar relation without damping term was obtained previously [14]. If $n = m$ is selected in Eq. (47), the special case of principle parametric resonances discussed in Section 3.2.1 will be obtained.

4.3. Difference-type combination resonances only ($\Omega + \omega_m \cong \omega_n$, $\Omega - \omega_m \ncong \omega_n$)

The corresponding complex amplitude evolution equations are given in Section 3.3 (i.e. Eqs. (27) and (28)). The complex amplitudes are transformed for convenience first

$$A_n = B_n\,\mathrm{e}^{\mathrm{i}\sigma T_1/2}, \quad A_m = B_m\,\mathrm{e}^{-\mathrm{i}\sigma T_1/2}. \tag{48}$$

Inserting them into Eqs. (27) and (28) yields

$$D_1 B_n + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma}{2}\right) B_n + k_{5mn} B_m = 0,$$
$$D_1 B_m + \left(\frac{\mu}{2} - \mathrm{i}\frac{\sigma}{2}\right) B_m + k_{6mn} B_n = 0. \tag{49}$$

One may now introduce the below forms for B_n and B_m

$$B_n = b_n \mathrm{e}^{\lambda T_1}, \quad B_m = b_m \mathrm{e}^{\lambda T_1} \tag{50}$$

and substituting into Eq. (49), one gets the following matrix equation:

$$\begin{bmatrix} \lambda + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma}{2}\right) & k_{5mn} \\ k_{6mn} & \lambda + \left(\frac{\mu}{2} - \mathrm{i}\frac{\sigma}{2}\right) \end{bmatrix} \begin{bmatrix} b_n \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{51}$$

For non-trivial solutions

$$\lambda_{1,2} = -\tfrac{1}{2}\mu \mp \tfrac{1}{2}\sqrt{-\sigma^2 + 4k_{5mn}k_{6mn}}. \tag{52}$$

It turns out that $k_{5mn}k_{6mn}$ is always a negative real number. Therefore solutions are stable upto order ε. This conclusion is also valid for the special case of $m = n$ discussed in Section 3.3.1.

4.4. Sum- and difference-type combination resonances together ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_p \cong \omega_q$) ($m \neq n \neq p \neq q$)

As outlined in Section 3.4, in the more general case, modes that are involved in sum- and difference-type resonances are uncoupled from each other. Therefore results of previous sections (i.e. Sections 4.2 and 4.3) are directly applicable, such that one does not expect any instability due to difference-type resonances but may encounter instability due to sum-type resonances upto order ε. Special cases of this section are however important and requires a distinct stability analysis. Depending on the numerical values of natural frequencies and fluctuation frequencies, resonances involving more than four modes may also be possible which are not treated in this work.

4.4.1. $n = q$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_p \cong \omega_n$)

As discussed in Section 3.4.1 this choice leads to negative frequencies and is impossible.

4.4.2. $m = q$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_p \cong \omega_m$)

This choice also leads to negative frequencies and is impossible.

4.4.3. $n = p$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_n \cong \omega_q$)

The equations for this case is given in Section 3.4.3 (i.e. Eq. (37)). A transformation of the below form makes simplifications

$$A_n = B_n e^{i\sigma_1 T_1/2}, \quad A_m = B_m e^{i\sigma_1 T_1/2}, \quad A_q = B_q e^{i\sigma_1 T_1/2} e^{i\sigma_2 T_1}. \tag{53}$$

Substituting into Eq. (38), one has

$$\begin{aligned} &D_1 B_n + \left(\frac{\mu}{2} + i\frac{\sigma_1}{2}\right) B_n + k_{3mn}\bar{B}_m + k_{6nq}B_q = 0, \\ &D_1 B_m + \left(\frac{\mu}{2} + i\frac{\sigma_1}{2}\right) B_m + k_{3nm}\bar{B}_n = 0, \\ &D_1 B_q + \left(\frac{\mu}{2} + i\left(\frac{\sigma_1}{2} + \sigma_2\right)\right) B_q + k_{5nq}B_n = 0. \end{aligned} \tag{54}$$

Defining the new variables

$$B_n = (b_{nR} + i b_{nI})e^{\lambda T_1}, \quad B_m = (b_{mR} + i b_{mI})e^{\lambda T_1}, \quad B_q = (b_{qR} + i b_{qI})e^{\lambda T_1} \tag{55}$$

and substituting into Eq. (54), one gets the following matrix equation:

$$\begin{bmatrix} \lambda + \frac{\mu}{2} & -\frac{\sigma_1}{2} & k_{3mnR} & k_{3mnI} & k_{6nqR} & -k_{6nqI} \\ \frac{\sigma_1}{2} & \lambda + \frac{\mu}{2} & k_{3mnI} & -k_{3mnR} & k_{6nqI} & k_{6nqR} \\ k_{3nmR} & k_{3nmI} & \lambda + \frac{\mu}{2} & -\frac{\sigma_1}{2} & 0 & 0 \\ k_{3nmI} & -k_{3nmR} & \frac{\sigma_1}{2} & \lambda + \frac{\mu}{2} & 0 & 0 \\ k_{5nqR} & -k_{5nqI} & 0 & 0 & \lambda + \frac{\mu}{2} & -\left(\frac{\sigma_1}{2} + \sigma_2\right) \\ k_{5nqI} & k_{5nqR} & 0 & 0 & \frac{\sigma_1}{2} + \sigma_2 & \lambda + \frac{\mu}{2} \end{bmatrix} \begin{bmatrix} b_{nR} \\ b_{nI} \\ b_{mR} \\ b_{mI} \\ b_{qR} \\ b_{qI} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{56}$$

Eigenvalues can be determined from the determinant of the coefficient matrix.

4.4.4. $m=p$ case ($\Omega-\omega_m \cong \omega_n$, $\Omega+\omega_m \cong \omega_q$)

As mentioned previously in Section 3.4.4, this case is similar to the case given in Section 4.4.3 if m and n are interchanged. Therefore, a separate treatment is unnecessary.

4.4.5. $m=n$ case ($\Omega-\omega_n \cong \omega_n$, $\Omega+\omega_p \cong \omega_q$)

This case is treated in Section 3.4.5 ($\Omega \cong 2\omega_n$, $\Omega \cong \omega_q - \omega_p$). Referring to Eq. (40), one realizes that difference-type combination resonances are uncoupled from the parametric resonance mode ω_n. As discussed previously, combination resonances do not yield instability upto order ε. Therefore, the stability of the system depends mainly upon the principal parametric resonances.

Inserting the below transformation to the first of Eq. (39)

$$A_n = B_n \, e^{i\sigma_1 T_1/2} \tag{57}$$

one has

$$D_1 B_n + \left(\frac{\mu}{2} + i\frac{\sigma_1}{2}\right) B_n + k_{3nn} \overline{B}_n = 0. \tag{58}$$

Defining the new variable as

$$B_n = (b_{nR} + i b_{nI}) e^{\lambda T_1} \tag{59}$$

and substituting into Eq. (58), one has the following matrix equation:

$$\begin{bmatrix} \lambda + \frac{\mu}{2} + k_{3nnR} & -\frac{\sigma_1}{2} + k_{3nnI} \\ \frac{\sigma_1}{2} + k_{3nnI} & \lambda + \frac{\mu}{2} - k_{3nnR} \end{bmatrix} \begin{bmatrix} b_{nR} \\ b_{nI} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{60}$$

The eigenvalues are

$$\lambda_{1,2} = -\frac{1}{2}\mu \mp \sqrt{-\frac{\sigma_1^2}{4} + k_{3nnR}^2 + k_{3nnI}^2}. \tag{61}$$

From here, the detuning parameter is calculated as follows:

$$\sigma_1 = \mp 2\sqrt{-\frac{\mu^2}{4} + k_{3nnR}^2 + k_{3nnI}^2}. \tag{62}$$

Then the stability boundaries are

$$\Omega = 2\omega_n \mp 2\varepsilon\sqrt{-\frac{\mu^2}{4} + k_{3nnR}^2 + k_{3nnI}^2}. \tag{63}$$

Note that the above result can be retrieved as a special case from Section 4.2 by taking $m=n$ in Eq. (47).

4.4.6. $p=q$ case ($\Omega+\omega_p \cong \omega_p$, $\Omega-\omega_m \cong \omega_n$)

As mentioned earlier in Section 3.4.6, both conditions $\Omega \cong 0$, $\Omega \cong \omega_n + \omega_m$ cannot simultaneously occur and hence discarded.

4.4.7. $n=p=m$ case ($\Omega-\omega_n \cong \omega_n$, $\Omega+\omega_n \cong \omega_q$)

This case is treated in Section 3.4.7. The relevant equations are given in Eq. (40). In this case, one mode is involved in a principal parametric resonance and the principal parametric resonance mode is involved further in a difference type combination resonance ($\Omega \cong 2\omega_n$, $\Omega \cong \omega_q - \omega_n$).

Introducing the transformations

$$A_n = B_n e^{i\sigma_1 T_1/2}, \quad A_q = B_q e^{i((\sigma_1/2)+\sigma_2)T_1} \tag{64}$$

and substituting into Eq. (40), one has

$$D_1 B_n + \left(\frac{\mu}{2} + \mathrm{i}\frac{\sigma_1}{2}\right) B_n + k_{3nn}\overline{B}_n + k_{6nq} B_q = 0,$$
$$D_1 B_q + \left[\frac{\mu}{2} + \mathrm{i}\left(\frac{\sigma_1}{2} + \sigma_2\right)\right] B_q + k_{5nq} B_n = 0. \tag{65}$$

The new variables can be expressed as follows:

$$B_n = (b_{nR} + \mathrm{i} b_{nI})\mathrm{e}^{\lambda T_1}, \quad B_q = (b_{qR} + \mathrm{i} b_{qI})\mathrm{e}^{\lambda T_1}. \tag{66}$$

Substituting them into Eq. (65), one gets the following matrix equation:

$$\begin{bmatrix} \lambda + \frac{\mu}{2} + k_{3nnR} & -\frac{\sigma_1}{2} + k_{3nnI} & k_{6nqR} & -k_{6nqI} \\ \frac{\sigma_1}{2} + k_{3nnI} & \lambda + \frac{\mu}{2} - k_{3nnR} & k_{6nqI} & k_{6nqR} \\ k_{5nqR} & -k_{5nqI} & \lambda + \frac{\mu}{2} & -\left(\frac{\sigma_1}{2} + \sigma_2\right) \\ k_{5nqI} & k_{5nqR} & \frac{\sigma_1}{2} + \sigma_2 & \lambda + \frac{\mu}{2} \end{bmatrix} \begin{bmatrix} b_{nR} \\ b_{nI} \\ b_{qR} \\ b_{qI} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{67}$$

Eigenvalues can be evaluated from the determinant of the coefficient matrix.

5. Numerical solutions

In this section numerical solutions will be presented. The velocity dependent frequencies are shown in Figs. 2 and 3 for the first and second modes, respectively for different flexural stiffness values. The frequencies decrease with increasing mean velocity. As the flexural rigidity increases (v_f), the frequencies increase also. Figs. 4–6 denote the first five modes for three different flexural stiffness values ($v_f = 0.2$, 0.6, 1.0). The dashed lines denote the complex frequency values, i.e. non-zero imaginary parts. The beam is unstable at those velocities.

In Ref. [26] analytical frequency equations were given in the special case of vanishing mean velocity. Our numerically calculated frequency results for $v_0 \to 0$ can directly be compared to those given in Ref. [26]. In that

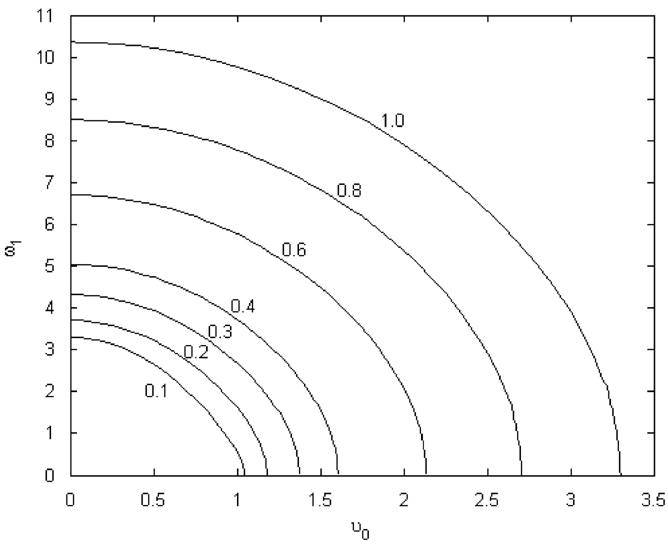

Fig. 2. Velocity dependent first mode frequencies for different flexural stiffness values (v_f values indicated on the figure).

reference, dimensional frequencies are given in Eq. (16) as follows:

$$\omega_n^{*2} = \left(\frac{cn\pi}{L}\right)^2 + \delta\left(\frac{n\pi}{L}\right)^4, \quad n = 1, 2, 3\ldots, \tag{68}$$

where c and δ are defined in that work as

$$c = \sqrt{\frac{P}{\rho A}}, \quad \delta = \frac{EI}{\rho A}. \tag{69}$$

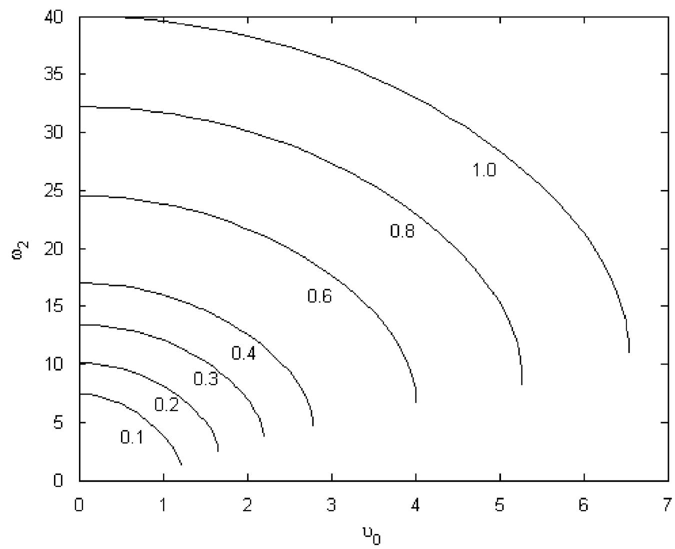

Fig. 3. Velocity dependent second mode frequencies for different flexural stiffness values (v_f values indicated on the figure).

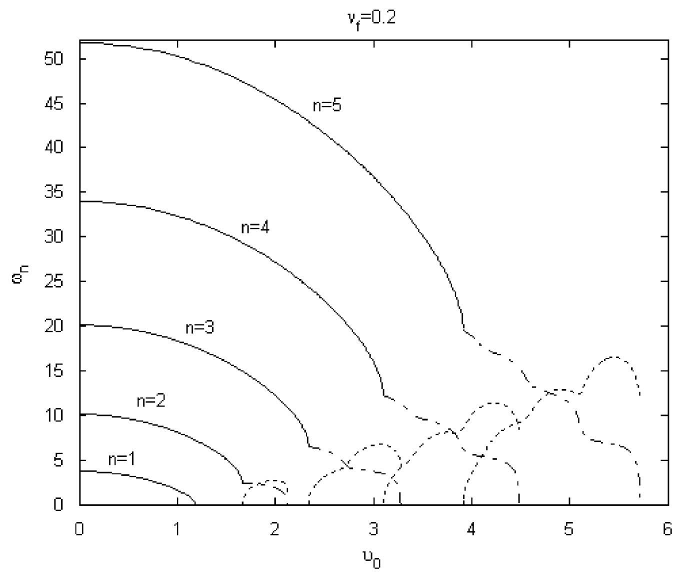

Fig. 4. Velocity dependent first five frequencies for flexural stiffness value $v_f = 0.2$.

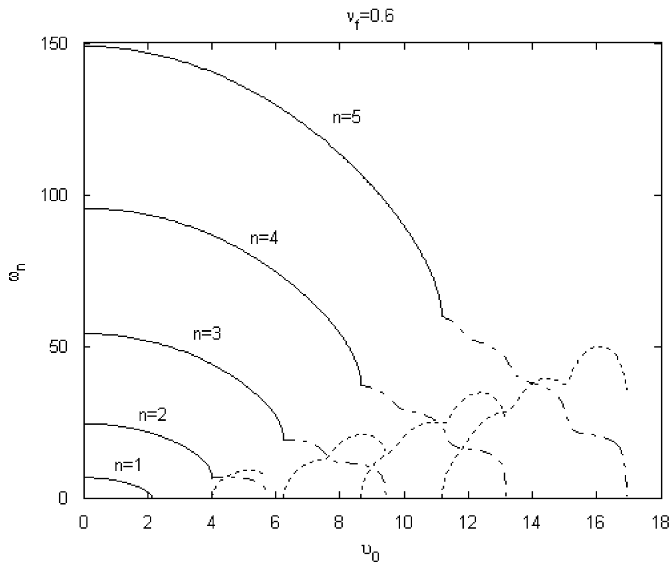

Fig. 5. Velocity dependent first five frequencies for flexural stiffness value $v_f = 0.6$.

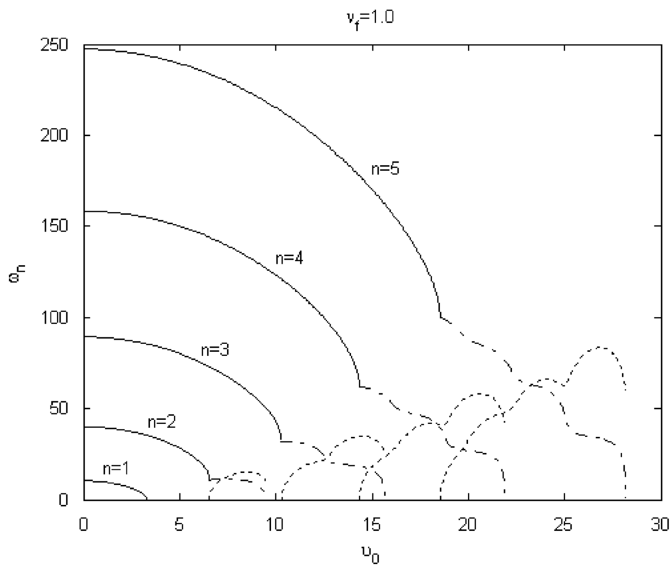

Fig. 6. Velocity dependent first five frequencies for flexural stiffness value $v_f = 1.0$.

The dimensional (denoted by asteriks) and dimensionless frequencies are related to each other as follows:

$$\omega^* = \omega\sqrt{\frac{P}{\rho A L^2}}. \tag{70}$$

Dividing Eq. (68) by $\sqrt{P/\rho AL^2}$, inserting the definitions (69) yields finally the dimesionless frequencies of Ref. [26] in terms of v_f parameter used in this work

$$\omega_n^2 = n^2\pi^2 + v_f^2 n^4 \pi^4, \quad n = 1, 2, 3 \ldots, \tag{71}$$

where v_f is given in Eq. (2). Our numerical frequency results for vanishing mean velocity are compared with the frequencies calculated by the analytical expression given in Eq. (71) in Table 1. Excellent match is observed in both results.

Possible combination resonances of sum and difference types are given in Fig. 7 for $v_f = 0.2$, in Fig. 8 for $v_f = 0.6$, and in Fig. 9 for $v_f = 1.0$. The first five modes are considered. As can be seen from the figures, sum of two modes and difference of another two modes may be close (or even equal) to each other for some specific values of the mean velocity. For a given mean velocity v_0 and fluctuation frequency Ω (y-axis), if the system is in the vicinity of an intersection point of two curves, one might expect resonances involving four modes. Resonances involving higher number of modes cannot be detected from the figures because only the first five modes and their combinations are calculated. It might happen that more than two curves intersect at a point or become very close to each other at a specific point indicating resonances involving higher number of modes. However, if Ω is kept small enough, then one can ensure that there is only one sum-type combination resonance and any possible difference resonance that involves higher modes and close to Ω will not alter the stability of the system because one knows that difference-type resonances do not yield instability upto order ε

Table 1
Comparison of the first three-dimensionless frequencies of [26] with the ones calculated in the present work for vanishing mean velocity case

v_f	ω_1 [26]	ω_1 (this study)	ω_2 [26]	ω_2 (this study)	ω_3 [26]	ω_3 (this study)
0.2	3.7103	3.7103	10.0906	10.0906	20.1105	20.1105
0.6	6.7035	6.7035	24.5062	24.5062	54.1228	54.1228
1.0	10.3575	10.3575	39.9753	39.9753	89.3250	89.3250

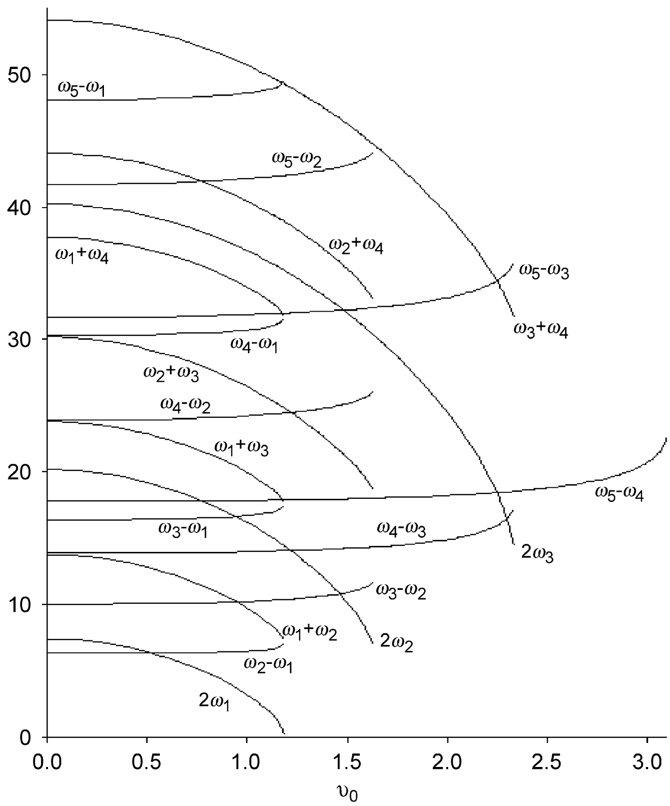

Fig. 7. Sum and difference of the first five natural frequencies and their variation with the mean velocity for $v_f = 0.2$.

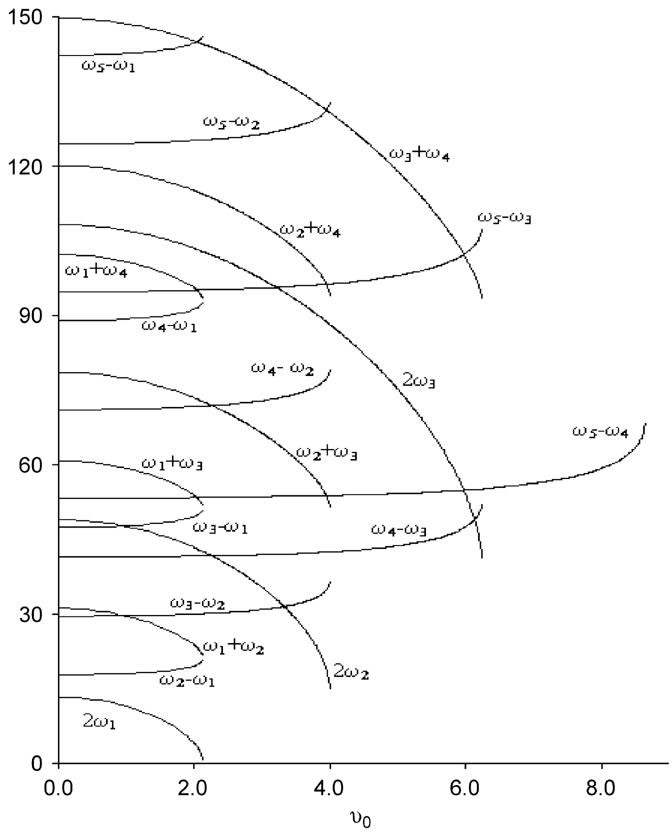

Fig. 8. Sum and difference of the first five natural frequencies and their variation with the mean velocity $v_f = 0.6$.

and furthermore they are uncoupled from the sum-type resonances. Therefore resonances involving four modes would be a reasonable assumption. Sample values can be deducted from the figures corresponding to different cases considered previously in the stability analysis.

The stability branches separating stable and unstable regions for only sum-type resonances are given in Figs. 10–13. The stability of this case is developed in Section 4.2 and the branches are drawn based on Eq. (47). In Fig. 10 the stability borders are shown for $v_f = 0.2$, $\upsilon_0 = 1.0$ and $\Omega = \omega_3 + \omega_1 + \varepsilon\sigma$, i.e. the velocity fluctuation frequency is close to the sum of the first and third frequencies. Note that from Fig. 7, there is no other curve close to $\omega_1 + \omega_3$ for $\upsilon_0 = 1.0$. Resonances occur due to only two modes. The region between the lines are unstable and the region outside the lines are stable. As expected, increasing damping shifts the stability borders upwards, that is instability is encountered at higher axial velocity fluctuation amplitudes ($\varepsilon\upsilon_1$). As the velocity fluctuation amplitude increases, the stability region widens. In Fig. 11 the first principal parametric resonance is shown with $\Omega = 2\omega_1 + \varepsilon\sigma$, i.e. the velocity fluctuation frequency is close to twice of the first frequency (choose $m = n = 1$ in Eq. (47)). Again one has to check from Fig. 7 that for the given $\upsilon_0 = 1$ value, there is no other interactions. Similar qualitative behavior is observed but damping has less effect compared to Fig. 10. In Fig. 12, the first and third mode sum-type combination resonance is considered for a different mean velocity and flexural rigidity. For the same mean velocity and flexural rigidity, the first principal parametric resonance is considered in Fig. 13. The effect of damping is again less compared to the sum-type resonance having the same system values.

For higher number of mode interactions, some numerical examples from Section 4.4 are presented. In Tables 2a–j example resonances from Section 4.4.3 when $\Omega - \omega_m \cong \omega_n$ and $\Omega + \omega_n = \omega_q$ are given (Section 4.4.4. is similar to Section 4.4.3 when n and m are interchanged). The resonance cases are identified from Figs. 7 to 9 first. Then the stability equations presented in Section 4.4.3 are used to determine stability. There different dimensionless stiffness values namely $v_f = 0.2$, 0.6 and 1.0 are used in the calculations. For a

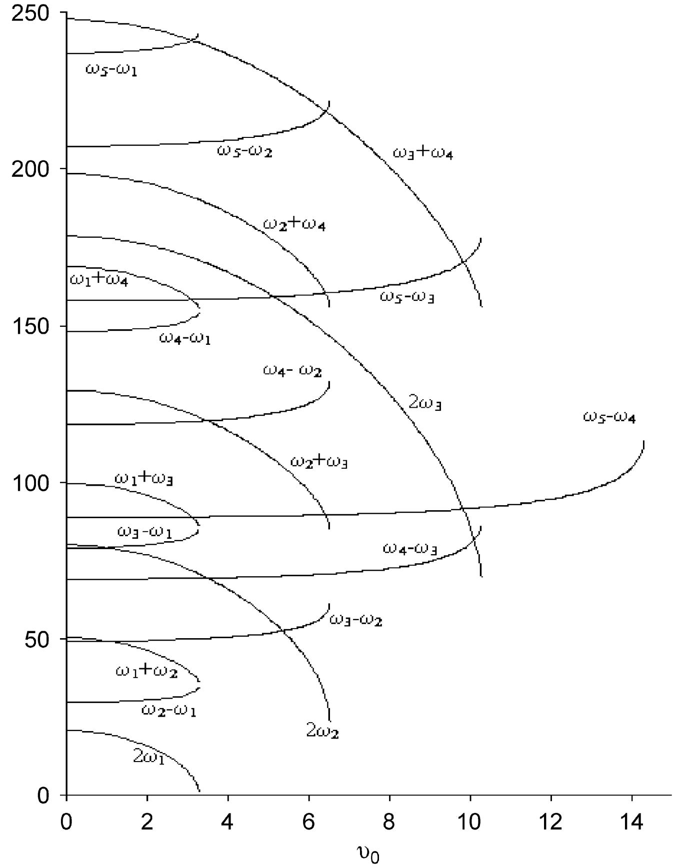

Fig. 9. Sum and difference of the first five natural frequencies and their variation with the mean velocity $v_f = 1.0$.

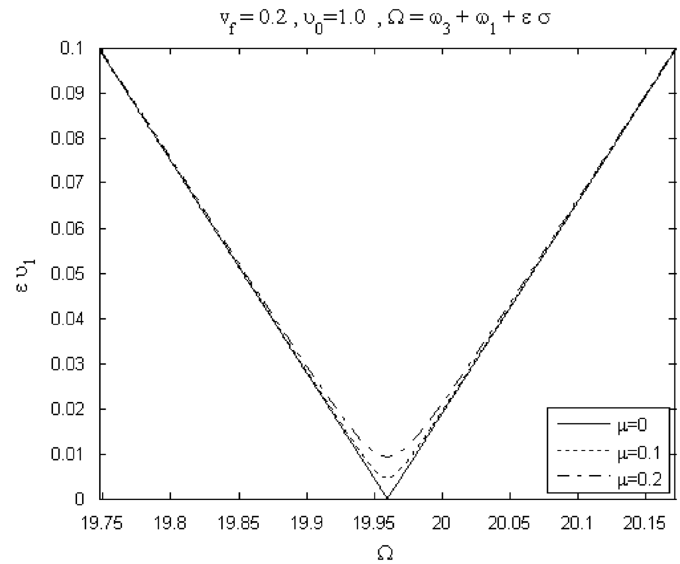

Fig. 10. Stable and unstable regions for sum-type combination resonances for the first and third modes ($v_f = 0.2$, $\upsilon_0 = 1.0$, $\mu = 0$ —, $\mu = 0.1$ - - -, $\mu = 0.2$ -.-.-).

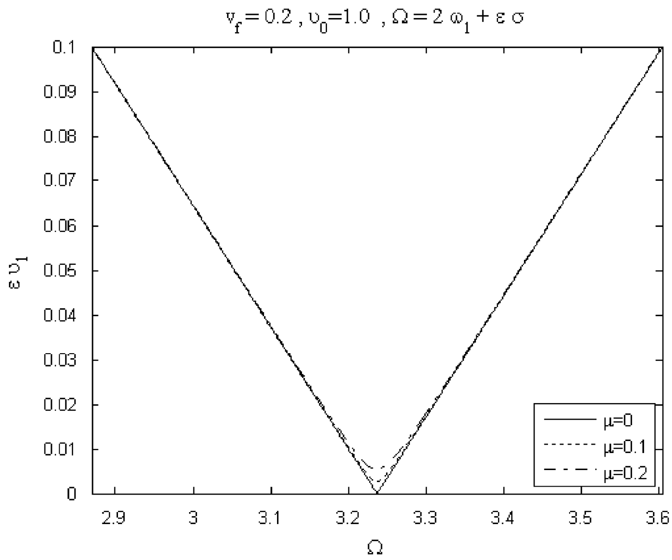

Fig. 11. Stable and unstable regions for principal parametric resonances for the first mode ($v_f = 0.2$, $\upsilon_0 = 1.1$, $\mu = 0$ —, $\mu = 0.1$ - - -, $\mu = 0.2$ -.-.-).

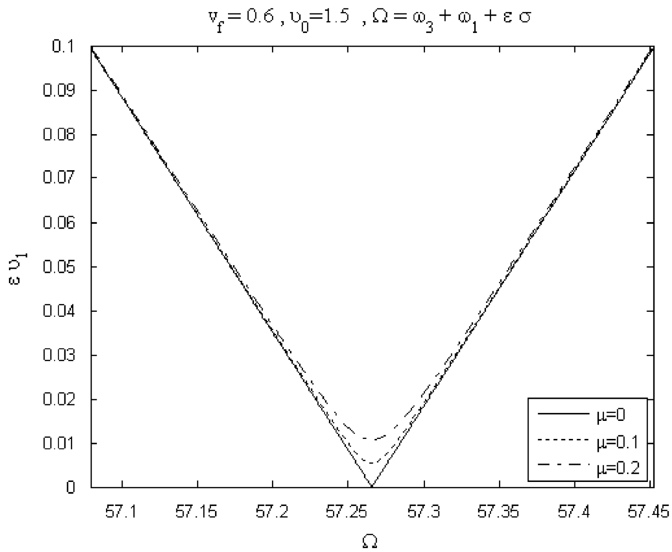

Fig. 12. Stable and unstable regions for sum-type combination resonances for the first and third modes ($v_f = 0.6$, $\upsilon_0 = 1.5$, $\mu = 0$ —, $\mu = 0.1$ - - -, $\mu = 0.2$ -.-.-).

selected parameter range, the amplitude of velocity fluctuations (εv_1) are increased and the stability of each point is calculated. For some of the cases, all points evaluated lead to stable solutions. However, for some others, instability starts after a specific value of the amplitude of fluctuations. Higher amplitude values might be needed to track instability for the tables of "all stable" solutions. Specific values of the modes for which resonance occurs are treated in Table 2. For example there exists a resonance involving second, fourth and fifth modes for $\Omega - \omega_4 \cong \omega_2$ and $\Omega + \omega_2 \cong \omega_5$ for $v_f = 0.2$, $\upsilon_0 = 0.776$, $\Omega = 41.925$ in Table 2a. Since the eigenvalues are complex, the solution is stable for εv_1 values ranging from 0 to 0.1. The resonances for some other sample frequencies are presented in Table 2b–j. As can be seen from the tables, unstable solutions appear at relatively low amplitude of fluctuation frequencies for some cases whereas they are undetectable for other cases for the parameter range considered.

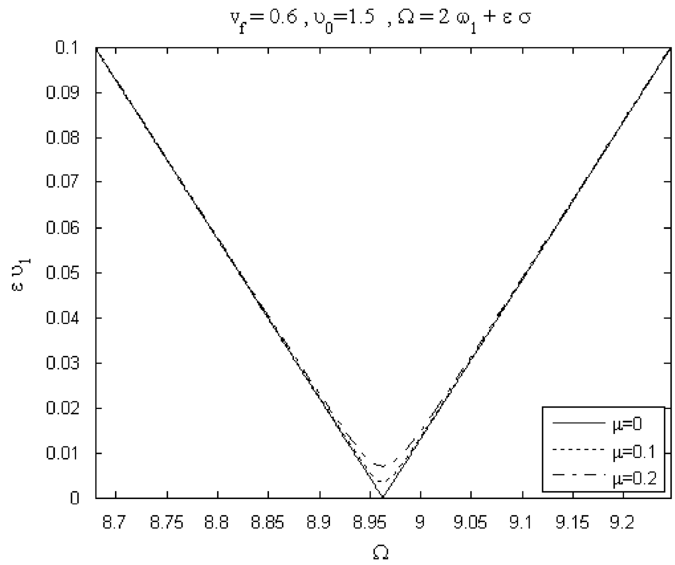

Fig. 13. Stable and unstable regions for principal parametric resonances for the first mode ($v_f = 0.6$, $\upsilon_0 = 1.5$, $\mu = 0$ —, $\mu = 0.1$ - - -, $\mu = 0.2$ -.-.-).

Table 2
Resonance conditions of Section 4.4.3

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
(a) Resonance conditions of Section 4.4.3 when $\Omega - \omega_4 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_5$, $v_f = 0.2$, $\upsilon_0 = 0.776$ ($\Omega = 41.925$, $\sigma_1 = 0.0290$, $\sigma_2 = -0.0626$, $\omega_2 = 8.9162$, $\omega_4 = 33.0059$, $\omega_5 = 50.8474$)											
εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S
(b) Resonance conditions of Section 4.4.3 when $\Omega - \omega_4 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 0.2$, $\upsilon_0 = 2.248$ ($\Omega = 34.43$, $\sigma_1 = 0.0971$, $\sigma_2 = -0.0824$, $\omega_3 = 9.2215$, $\omega_4 = 25.1988$, $\omega_5 = 43.6598$)											
εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S
(c) Resonance conditions of Section 4.4.3 when $\Omega - \omega_3 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_4$, $v_f = 0.2$, $\upsilon_0 = 1.218$ ($\Omega = 24.495$, $\sigma_1 = 0.0520$, $\sigma_2 = -0.0976$, $\omega_2 = 7.0362$, $\omega_3 = 17.4536$, $\omega_4 = 31.5410$)											
εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	U	U
(d) Resonance conditions of Section 4.4.3 when $\Omega - \omega_1 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 0.2$, $\upsilon_0 = 0.94$ ($\Omega = 10.222$, $\sigma_1 = -0.1287$, $\sigma_2 = 0.1189$, $\omega_1 = 1.8939$, $\omega_2 = 8.3410$, $\omega_3 = 18.5511$)											
εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S
(e) Resonance conditions of Section 4.4.3 when $\Omega - \omega_4 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 0.6$, $\upsilon_0 = 5.97$ ($\Omega = 102.27$, $\sigma_1 = -0.0770$, $\sigma_2 = 0.0623$, $\omega_3 = 27.4702$, $\omega_4 = 74.8075$, $\omega_5 = 129.7340$)											
εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

Table 2 (*continued*)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10

(f) Resonance conditions of Section 4.4.3 when $\Omega - \omega_3 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_4$, $v_f = 0.6$, $\upsilon_0 = 2.26$ ($\Omega = 71.907$, $\sigma_1 = -0.0811$, $\sigma_2 = 0.0712$, $\omega_2 = 20.8591$, $\omega_3 = 51.0560$, $\omega_4 = 92.7590$)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(g) Resonance conditions of Section 4.4.3 when $\Omega - \omega_1 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 0.6$, $\upsilon_0 = 0.97$ ($\Omega = 29.685$, $\sigma_1 = -0.0567$, $\sigma_2 = -0.1327$, $\omega_1 = 5.8237$, $\omega_2 = 23.8670$, $\omega_3 = 53.5653$)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	U	U	U	U	U	U	U

(h) Resonance conditions of Section 4.4.3 when $\Omega - \omega_4 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 1.0$, $\upsilon_0 = 9.836$ ($\Omega = 170.31$, $\sigma_1 = 0.0597$, $\sigma_2 = -0.0507$, $\omega_3 = 45.7413$, $\omega_4 = 124.5628$, $\omega_5 = 216.0563$)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(i) Resonance conditions of Section 4.4.3 when $\Omega - \omega_3 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_4$, $v_f = 1.0$, $\upsilon_0 = 3.5$ ($\Omega = 119.631$, $\sigma_1 = 0.1036$, $\sigma_2 = -0.1118$, $\omega_2 = 34.7124$, $\omega_3 = 84.9082$, $\omega_4 = 154.3546$)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(j) Resonance conditions of Section 4.4.3 when $\Omega - \omega_1 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 1.0$, $\upsilon_0 = 0.98$ ($\Omega = 49.394$, $\sigma_1 = 0.1341$, $\sigma_2 = -0.0605$, $\omega_1 = 9.7974$, $\omega_2 = 39.5832$, $\omega_3 = 88.9832$)

εv_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	U	U	U	U	U	U

Table 3
Resonance conditions of Section 4.4.7

$\varepsilon \mathrm{v}_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10

(a) Resonance conditions of Section 4.4.7 when $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 0.2$, $\upsilon_0 = 1.48$ ($\Omega = 32.222$, $\sigma_1 = 0.0484$, $\sigma_2 = -0.0445$, $\omega_3 = 16.1086$, $\omega_5 = 48.3350$)

$\varepsilon \mathrm{v}_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(b) Resonance conditions of Section 4.4.7 when, $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_4$, $v_f = 0.2$, $\upsilon_0 = 2.298$ ($\Omega = 16.506$, $\sigma_1 = 0.0983$, $\sigma_2 = -0.0858$, $\omega_3 = 8.2481$, $\omega_4 = 24.7627$)

$\varepsilon \mathrm{v}_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	U	U	U	U	U	U	U	U

(c) Resonance conditions of Section 4.4.7 when $\Omega - \omega_2 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 0.2$, $\upsilon_0 = 1.46$ ($\Omega = 10.818$, $\sigma_1 = -0.0587$, $\sigma_2 = 0.0697$, $\omega_2 = 5.4119$, $\omega_3 = 16.2230$)

$\varepsilon \mathrm{v}_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

Table 3 (*continued*)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10

(d) Resonance conditions of Section 4.4.7 when $\Omega - \omega_1 \cong \omega_1$, $\Omega + \omega_1 \cong \omega_2$, $v_f = 0.2$, $\upsilon_0 = 0.509$ ($\Omega = 6.398$, $\sigma_1 = -0.0824$, $\sigma_2 = 0.0742$, $\omega_1 = 3.2031$, $\omega_2 = 9.5937$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(e) Resonance conditions of Section 4.4.7 when $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 0.6$, $\upsilon_0 = 3.21$ ($\Omega = 95.636$, $\sigma_1 = 0.0221$, $\sigma_2 = -0.0255$, $\omega_3 = 47.8169$, $\omega_5 = 143.4554$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	S

(f) Resonance conditions of Section 4.4.7 when $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_4$, $v_f = 0.6$, $\upsilon_0 = 6.129$ ($\Omega = 49.057$, $\sigma_1 = -0.0267$, $\sigma_2 = 0.0177$, $\omega_3 = 24.5298$, $\omega_4 = 73.5851$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	U	U	U	U	U	U	U	U

(g) Resonance conditions of Section 4.4.7 when $\Omega - \omega_2 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 0.6$, $\upsilon_0 = 3.322$ ($\Omega = 31.571$, $\sigma_1 = 0.0365$, $\sigma_2 = 0.0540$, $\omega_2 = 17.7837$, $\omega_3 = 47.3493$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	U

(h) Resonance conditions of Section 4.4.7 when $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_5$, $v_f = 1.0$, $\upsilon_0 = 5.136$ ($\Omega = 159.271$, $\sigma_1 = 0.0437$, $\sigma_2 = -0.0924$, $\omega_3 = 79.6333$, $\omega_5 = 238.9136$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	S	S	S	U

(i) Resonance conditions of Section 4.4.7 when $\Omega - \omega_3 \cong \omega_3$, $\Omega + \omega_3 \cong \omega_4$, $v_f = 1.0$, $\upsilon_0 = 10.1022$ ($\Omega = 81.696$, $\sigma_1 = -0.0531$, $\sigma_2 = 0.0647$, $\omega_3 = 40.8507$, $\omega_5 = 122.5402$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	U	U	U	U	U	U	U	U

(j) Resonance conditions of Section 4.4.7 when $\Omega - \omega_2 \cong \omega_2$, $\Omega + \omega_2 \cong \omega_3$, $v_f = 1.0$, $\upsilon_0 = 5.362$ ($\Omega = 52.481$, $\sigma_1 = -0.0272$, $\sigma_2 = -0.0196$, $\omega_2 = 26.2419$, $\omega_3 = 78.7248$)

εv$_1$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Stable/unstable	S	S	S	S	S	S	S	U	U	U	U

The possible cases for the resonance type $\Omega - \omega_n \cong \omega_n$ and $\Omega + \omega_n \cong \omega_q$ are given in Table 3a–j (Section 4.4.7) for different v_f values. For example there is a resonance involving third and fifth modes for $\Omega - \omega_3 \cong \omega_3$ and $\Omega + \omega_3 \cong \omega_5$ for $v_f = 0.2$, $\upsilon_0 = 1.48$, $\Omega = 32.222$ in Table 3a which do not lead to instability. Instabilities are detected in Tables 3b,f–j choices but the rest do not lead to instability for the parameter range considered.

6. Concluding remarks

In this study, transverse vibrations of an axially traveling Euler–Bernoulli beam are investigated. The simply supported beam is traveling with a velocity varying harmonically about a constant mean value with small fluctuation amplitudes. The method of Multiple Scales is applied to the equation of motion in search of infinite mode approximate solutions. The formalism for detecting arbitrary number of modes involved in a resonance is developed first. Sum- and difference-type combination resonances and upto four modes involved in such resonances are considered. Various cases and their stability are examined and the necessary equations for stability are derived. Stability borders are drawn for resonances due to sum-type combination of two modes. Damping shifts the unstable regions upwards and amplitude of excitation widens the unstable regions. For higher number of mode resonances (i.e. resonances involving upto four modes), stability results are presented for some sample values in the tables. In some of the cases considered, increase in mean velocity fluctuation amplitude causes instability.

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